

RIEMANNIAN POLYHEDRA AND LIOUVILLE-TYPE THEOREMS FOR HARMONIC MAPS

ZAHRA SINA EI

ABSTRACT. This paper is a study of harmonic maps from Riemannian polyhedra to (locally) non-positively curved geodesic spaces in the sense of Alexandrov. We prove Liouville-type theorems for subharmonic functions and harmonic maps under two different assumptions on the source space. First we prove the analogue of the Schoen-Yau Theorem on a complete (smooth) pseudomanifolds with non-negative Ricci curvature. Then we study 2-parabolic admissible Riemannian polyhedra and prove some vanishing results on them.

1. INTRODUCTION

Harmonic maps between singular spaces have received considerable attention since the early 1990s. Existence of energy minimizing locally Lipschitz maps from Riemannian manifolds into Bruhat-Tits buildings and Corlette's version of Margulis's super-rigidity theorem were proved in [GS92]. In [KS93] Korevaar and Schoen constructed harmonic maps from domains in Riemannian manifolds into Hadamard spaces as a boundary value problem. The book [EF01] by Eells and Fuglede contains a description of the application of the methods of [KS93] to the study of maps between polyhedra, see also [Che95, DM08, DM10].

Our first objective in this paper is to prove Liouville-type theorems for harmonic maps. We prove the analogue of the Schoen-Yau Theorem on complete (smooth) pseudomanifolds with non-negative Ricci curvature. To this end, we generalize some Liouville-type theorems for subharmonic functions from [Yau82].

The classical Liouville theorem for functions on manifolds states that on a complete Riemannian manifold with non-negative Ricci curvature, any harmonic function bounded from one side must be a constant. In [Yau82], Yau proves that there is no non-constant, smooth, non-negative, L^p , $p > 1$, subharmonic function on a complete Riemannian manifold. He also proves that every continuous subharmonic function defined on a complete Riemannian manifold whose local Lipschitz constant is bounded by L^1 function is also harmonic. Furthermore if the L^1 function belongs to L^2 as well, and the manifold has non-negative Ricci curvature, then the subharmonic function is constant. In the smooth setting, there are two types of assumptions that

have been studied on the Liouville property of harmonic maps. One is the finiteness of the energy and the other is the smallness of the image. For example, Schoen and Yau [SY76] proved that any non-constant harmonic map from a complete manifold of non-negative Ricci curvature to a manifold of non-positive sectional curvature has infinite energy. Hildebrandt-Jost-Widman [HJW81] (see also [Hil82, Hil85]) proved a Liouville-type theorem for harmonic maps into regular geodesic (open) balls in a complete C^3 Riemannian manifold from a simple or compact C^1 Riemannian manifold. For more references for Liouville theorem for harmonic maps and functions in both smooth and singular setting see the introduction in [KS08].

A connected locally finite n -dimensional simplicial polyhedron X is called admissible, if X is dimensionally n -homogeneous and X is locally $(n-1)$ -chainable. It is called circuit if instead it is $(n-1)$ -chainable and every $(n-1)$ -simplex is the face of at most two n -simplex and pseudomanifold if it is admissible circuit. A polyhedron X becomes a Riemannian polyhedron when endowed with a Riemannian structure g , defined by giving on each maximal simplex s of X a Riemannian metric g (bounded measurable) equivalent to a Euclidean metric on s (see [EF01]).

There exist slightly different notions of boundedness of Ricci curvature from below on general metric spaces. See for example [Stu06, LV09, Oht07, KS01, KS03] and the references therein. In the following by $\text{Ric}_{N,\mu_g} \geq K$ we mean that (X, g, μ_g) satisfies the measure contraction property. This is the convention adopted in [Oht07, Stu06]. As this definition is somewhat technical we refer the reader to Section 3 for a precise statement.

The definition of harmonic maps from admissible Riemannian polyhedra to metric spaces is similar to the one in the smooth setting. However due to lack of smoothness some care is needed in defining the notions of energy density, the energy functional and energy minimizing maps. Precise definitions and related results can be found in Subsection 2.7.

We can state now the main results which we obtain in this direction.

Theorem 1.1. *Suppose (X, g) is a complete, admissible Riemannian polyhedron, and $f \in W_{\text{loc}}^{1,2}(X) \cap L^2(X)$ is a non-negative, weakly subharmonic function. Then f is constant.*

Theorem 1.2. *Let (X, g, μ_g) be a complete non-compact pseudomanifold. Let f be continuous, weakly subharmonic and belonging to $W_{\text{loc}}^{1,2}(X)$, such that $\|\nabla f\|_{L^1}$ is finite. Then f is a harmonic function.*

Theorem 1.3. *Let (X, g, μ_g) be a complete, smooth n -pseudomanifold. Suppose X has non-negative N -Ricci curvature, Ric_{N,μ_g} , for $N \geq n$. Let f be a continuous, weakly subharmonic function belonging to $W_{\text{loc}}^{1,2}(X)$ such that both $\|\nabla f\|_{L^1}$ and $\|\nabla f\|_{L^2}$ are finite. Then f is a constant function.*

Here by a smooth pseudomanifold we mean a simplexwise smooth, pseudomanifold which is smooth outside of its singular set. That situation arises when the space is a projective algebraic variety. The Fubini-Study metric

of projective space induces a Bergman metric out of singular set which is Kahler. The difficulty in extending existing results lie in the lack of a differentiable structure on admissible polyhedron in general, and the loss of completeness outside the singular set even in the case of smooth pseudomanifolds. Moreover the classical notion of Laplace operator doesn't exist in the non-smooth setting. To circumvent this latter problem, influenced by the work of [Gig12], we define the Laplacian of a subharmonic function as a unique Radon measure (Theorem 4.2). As a consequence we obtain Green's formula for such functions.

The following two corollaries are important consequences of theorems 1.1 and 1.3.

Corollary 1.4. *Let (X, g, μ_g) be a complete, smooth n -pseudomanifold. Suppose X has non-negative N -Ricci curvature, for $N \geq n$. Suppose Y is a Riemannian manifold of non-positive curvature, and $u : (X, g) \rightarrow (Y, h)$ a continuous harmonic map belonging to $W_{\text{loc}}^{1,2}(X, Y)$. If u has finite energy, then it is a constant map.*

Corollary 1.5. *Let (X, g, μ_g) be a complete, smooth n -pseudomanifold. Suppose X has non-negative N -Ricci curvature, $N \geq n$. Let Y be a simply connected, complete geodesic space of non-positive curvature and $u : (X, g) \rightarrow Y$ a continuous harmonic map with finite energy, belonging to $W_{\text{loc}}^{1,2}(X, Y)$. If $\int_M \sqrt{e(u)} d\mu_g < \infty$, then u is a constant map.*

Our second objective in this paper is the study of 2-parabolic admissible polyhedra. We say a connected domain Ω in an admissible Riemannian polyhedron is 2-parabolic, if for every compact sets, its relative capacity with respect to Ω is zero. Our main theorem is

Theorem 1.6. *Let X be 2-parabolic pseudomanifold. Let f in $W_{\text{loc}}^{1,2}(X)$ be a continuous, weakly subharmonic function, such that $\|\nabla f\|_{L^1}$ and $\|\nabla f\|_{L^2}$ are finite. Then f is constant.*

In order to prove this theorem we need to generalize some of the results in [Hol90]. This is done in Subsection 5.2. In particular we will need following propositions.

Proposition 1.7. *Let (X, g) be 2-parabolic admissible Riemannian polyhedron. Suppose f in $W_{\text{loc}}^{1,2}(X)$ is a positive, continuous superharmonic function on X . Then f is constant.*

Proposition 1.8. *Let X be 2-parabolic admissible Riemannian polyhedron. Let f in $W_{\text{loc}}^{1,2}(X)$ be a harmonic function such that $\|\nabla f\|_{L^2}$ is finite. Then f is constant.*

The proofs of the propositions above follow a similar pattern as their equivalents for Riemannian manifolds. They are based on the fact that admissible Riemannian polyhedra admits an exhaustion by regular domains, and the validity of comparison principle on admissible Riemannian polyhedra. The main new element in the proof of Theorem 1.9 is

Proposition 1.9. *Let X be a 2-parabolic pseudomanifold. Let f in $W_{\text{loc}}^{1,2}(X)$ be a continuous, weakly subharmonic function, such that $\|\nabla f\|_{L^1}$ and $\|\nabla f\|_{L^2}$ are finite. Then f is harmonic.*

Just as in the case of smooth pseudomanifolds Theorem 1.9 can be used to prove

Corollary 1.10. *Let (X, g) be a 2-parabolic pseudomanifold with g simplexwise smooth. Let Y be a simply connected complete geodesic space of non-positive curvature and $u : (X, g) \rightarrow Y$ a continuous harmonic map with finite energy belonging to $W_{\text{loc}}^{1,2}(X, Y)$. If we have $\int_X \sqrt{e(u)} d\mu_g < \infty$ then u is a constant map.*

Also by use of Proposition 1.7, we will obtain

Corollary 1.11. *Let (X, g) be a 2-parabolic admissible Riemannian polyhedron with simplexwise smooth metric g . Let Y be a complete geodesic space of non- positive curvature and $u : (X, g) \rightarrow Y$ a continuous harmonic map belonging to $W_{\text{loc}}^{1,2}(X, Y)$, with bounded image. Then u is a constant map.*

The rest of this paper is organized as follows. In Section 2, we give a complete background on Riemannian polyhedron and analysis on them. Most definition and results have been taken directly from [EF01]. In Subsection 2.2, we compare the L^2 based Sobolev space on admissible Riemannian polyhedra as in [EF01], with the one in [Che99], and show that they are equal. As we couldn't find references in the literature we provided a rather detailed explanation of this fact. In Section 3, we provide the definition of two notions of Ricci curvature- the curvature dimension condition $CD(K, N)$ and the measure contraction property $MCP(K, N)$ - on metric measure spaces. We show that both notions are applicable to Riemannian polyhedra. In Proposition 3.6 we show that any non-compact complete n -dimensional Riemannian polyhedron of non-negative Ricci curvature $MCP(0, N)$, $N \geq n$ has infinite volume. Subsection 4.1 is devoted to Theorems 1.1, 1.2, 1.3 and Subsection 4.2 to Theorems 1.4, and 1.5. In Section 5 we show that as in the smooth case the " approximation by 1" property holds on admissible 2-parabolic polyhedra (Lemma 5.2). Moreover, we prove that removing the singular set of a 2-parabolic pseudomanifold yields a 2-parabolic manifold (Lemma 5.3). The rest of this Section is the detailed proof of Theorem 1.9 and its corollaries.

ACKNOWLEDGEMENT

This work is part of my Ph.D. dissertation. I thank my advisor Professor Marc Troyanov for his guidance and support in the completion of this work.

2. PRELIMINARIES

2.1. Riemannian polyhedra. In this subsection we have gathered standard definitions and results about Riemannian polyhedra which will be used in the rest of the manuscript.

Definition 2.1. [EF01] *A countable locally finite simplicial complex K , consists of a countable set $\{v\}$ of elements, called vertices, and a set $\{s\}$ of finite non void subsets of vertices, called simplexes, such that*

- *any set consisting of exactly one vertex is a simplex,*
- *any non void subset of a simplex is a simplex,*
- *every vertex belongs to only finitely many simplexes (the local finiteness condition).*

To the simplicial complex K , we associate a metric space $|K|$ defined as follow. The space $|K|$ is the set of all formal finite linear combinations $\alpha = \sum_{v \in K} \alpha(v)v$ of vertices of K such that $0 \leq \alpha(v) \leq 1$, $\sum_{v \in K} \alpha(v) = 1$ and $\{v : \alpha(v) > 0\}$ is a simplex of K . $|K|$ is made into a metric space with barycentric distance $d(\alpha, \beta)$ between two points $\alpha = \sum \alpha(v)v$ and, $\beta = \sum \beta(v)v$ of $|K|$ given by the finite sum

$$d(\alpha, \beta)^2 = \sum_{v \in K} (\alpha(v) - \beta(v))^2.$$

With this metric $|K|$ is locally compact and separable.

Lemma 2.2. [EF01] *Let K be a countable, locally finite simplicial complex of finite dimension n , and V a Euclidean space of dimension $2n + 1$. There exists an affine Lipschitz homeomorphism f of $|K|$ onto a closed subset of V .*

Definition 2.3. [EF01] *We shall use the term polyhedron to mean a connected locally compact separable Hausdorff space X for which there exists a simplicial complex K and a homeomorphism θ of $|K|$ onto X . Any such pair $T = (K, \theta)$ is called a triangulation of X .*

Definition 2.4. [EF01] *A polyhedron X will be called admissible if in some (hence in any) triangulation,*

(i) *X is dimensionally homogeneous, i.e. all maximal simplexes have the same dimension $n (= \dim X)$, or equivalently every simplex is a face of some n -simplex and*

(ii) *X is locally $(n-1)$ -chainable, i.e. for every connected open set $U \subset X$, the open set $U \setminus X^{n-2}$ is connected. The boundary ∂X of a polyhedron X is the union of all non maximal simplexes contained in only one maximal simplex. In this paper we always assume that (X, g) satisfies $\partial X = \emptyset$.*

Definition 2.5. [EF01] *By an n -circuit we mean a polyhedron X of homogeneous dimension n such that in some, (and hence any) triangulation,*

(a) *every $(n-1)$ -simplex is a face of at most two n -simplexes (exactly two if $\partial X = 0$), and*

(b) X is $(n - 1)$ -chainable, i.e. $X \setminus X^{n-2}$ is connected, or equivalently any two n -simplexes can be joined by a chain of contiguous $(n - 1)$ - and n -simplexes.

Let $S = S(X)$ denote the singular set of an n -circuit X : the complement of the set of all points of X having a neighborhood which is a topological n -manifold (possibly with boundary). S is a closed triangulable subspace of X of codimension ≥ 2 , and $X \setminus S$ is a topological n -manifold dense in X (there is a PL stratification for such spaces). An admissible circuit is called a pseudomanifold.¹

Definition 2.6. [EF01] *A Lipschitz polyhedron is a metric space X which is the image of the metric space $|K|$ of some complex K under a Lipschitz homeomorphism $\theta : |K| \rightarrow X$. The pair (K, θ) is then called a Lipschitz triangulation (or briefly a triangulation) of the Lipschitz polyhedron X .*

By a null set in a Lipschitz polyhedron X is understood a set $Z \subset X$ such that Z meets every maximal simplex s (relative to some, and hence any triangulation $T = (K, \theta)$ of X) in a set whose pre-image under θ has p -dimensional Lebesgue measure 0, $p = \dim s$.

From Lemma 2.2 follows that every Lipschitz polyhedron (X, d_X) can be mapped Lipschitz homeomorphically and (simplexwise) affinely onto a closed subset of a Euclidean space. Analogously, let X denote a locally compact separable metric space, each point of which has a neighborhood that can be bi-Lipschitz embedded as a closed subset of some Euclidean space. Then there is a proper Lipschitz homeomorphism of X onto a closed subset of some Euclidean space. For a Lipschitz circuit X with singular set S , $X \setminus S$ is a Lipschitz manifold.

2.1.1. Riemannian Structure on a polyhedron. A Riemannian polyhedron $X = (X, g)$ will be defined as a Lipschitz polyhedron X endowed with a co-variant bounded measurable Riemannian metric tensor g_s on each maximal simplex s , satisfying the ellipticity condition 1 below.

Let $T = (K, \theta)$ be a specific (Lipschitz) triangulation of the Lipschitz polyhedron X . We shall view $|K|$ as embedded in a Euclidean space V via an affine Lipschitz homeomorphism. Suppose, to begin with, that X has homogeneous dimension n . Choose a measurable Riemannian metric g_s on the open Euclidean n -simplex $\theta^{-1}(s^\circ)$ of $|K| \subset V$. In terms of Euclidean coordinates x_1, \dots, x_n of points $x = \theta^{-1}(p) \in \theta^{-1}(s^\circ)$ (in its affine span in V), g_s thus assigns to almost every point $p \in s^\circ$, or to $x \in \theta^{-1}(s^\circ)$, an $n \times n$ symmetric positive definite matrix

$$g_s(x) = (g_{ij}(x))_{i,j=1,\dots,n}$$

¹In many texts the term pseudomanifold used for what we called a circuit.

with measurable real entries, and there is a constant $\Lambda_s > 0$ such that (with the usual summation convention)

$$\Lambda_s^{-2} \sum_{i=1}^n (\xi^i)^2 \leq g_{ij}^s(x) \xi^i \xi^j \leq \Lambda_s^2 \sum_{i=1}^n (\xi^i)^2 \quad (1)$$

for a.e. $x \in \theta^{-1}(s^\circ)$ and every $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. This condition is independent of the choice of Euclidean frame on $\theta^{-1}(s^\circ)$. The second inequality in 1 amounts to the components of g_s being bounded.

If X is not necessarily dimensionally homogeneous, define g , as above for each homogeneous subpolyhedron X_n , where X_n denotes the union of all maximal n -simplexes of X .

The covariantly defined map $g : s \rightarrow g_s$, thus defined on the set $S^*(X, T)$ of all maximal simplexes s of T , is called the Riemannian (polyhedral) structure, or metric, on X .

We require that, relative to a fixed triangulation T of a Riemannian polyhedron X , $\Lambda := \sup\{\Lambda_s : s \in S^*(X, T)\} < \infty$. Note that Λ here depends on the choice of constants λ_s . The smallest such constant Λ will be called the ellipticity constant of $X = (X, T, g)$.

A Riemannian metric g on a polyhedron X is said to be continuous, if relative to some (hence any) triangulation, g_s is continuous up to the boundary on each maximal simplex s , and for any two maximal simplexes s and s' sharing a face t , g_s and $g_{s'}$ induce the same Riemannian metric on t . There is a similar notion of a Lipschitz continuous Riemannian metric.

Definition 2.7. [EF01] *Let \mathcal{Z} denote the collection of all null sets $Z \subset X$. For a given triangulation $T = (K, \theta)$ consider in particular the set $Z_T \in \mathcal{Z}$ obtained from X by removing from each maximal simplex s in X those points of s° which are Lebesgue points for g_s (that is, for every component g). For $x, y \in X$ and any $Z \in \mathcal{Z}$ such that $Z \supset Z_T$ we denote by $\text{Lip}^Z(x, y, X)$ the family of all Lipschitz continuous paths $\gamma : [a, b] \rightarrow (X, e)$ with $\gamma(a) = x$, $\gamma(b) = y$ which are transversal to Z in the sense that $\gamma^{-1}(Z)$ is a null subset of $[a, b]$. Here e denotes the metric induced by the Euclidean distance on a Euclidean space V in which $|K|$ is affinely Lipschitz embedded. The length $L_T(\gamma)$ of such a path γ is well defined by*

$$L_T(\gamma) = \sum_{s \in S^*(X, T)} \int_{\gamma^{-1}(s^\circ)} \sqrt{(g_{ij}^s \circ \theta^{-1} \circ \gamma) \dot{\gamma}^i \dot{\gamma}^j}$$

where $(\gamma^1, \dots, \gamma^n) = \theta^{-1} \circ \gamma$ in terms of Euclidean coordinates on the open Euclidean simplex $\theta^{-1}(s^\circ)$, and the dot means differentiation. Write

$$\varrho_Z(x, y) = \inf_{\gamma} \{ L_T(\gamma) : \gamma \in \text{Lip}^Z(x, y, X) \},$$

Finally set

$$d_X(x, y) = \sup_Z \{ \varrho_Z(x, y) : Z \in \mathcal{Z}, Z \supset Z_T \}.$$

By the definition above,

$$\Lambda^{-1} \leq \frac{d_X(x, y)}{d_X^e(x, y)} \leq \Lambda \quad \text{for } x, y \in X,$$

where d_X^e is the distance defined as above for Euclidean Riemannian metric g^e on X .

Proposition 2.8. [EF01] *a) The distance function $d_X^g = d_X$ on a Riemannian polyhedron (X, g) is intrinsic, in particular independent of the chosen triangulation.*

(b) (X, d_X) is a length space (hence a geodesic space, if complete).

(c) d_X equals the Caratheodory distance

$$d_X(x, y) = \max\{ |u(x) - u(y)| : u \in \text{Lip}(X), |\nabla u| \leq 1 \text{ a.e. in } X \}.$$

Here $\text{Lip}(X)$ denotes the class of Lipschitz continuous functions $u : X \rightarrow \mathbb{R}$ and $|\nabla u|$ is the norm of the Riemannian gradient. The latter is defined a.e. in X (that is, a.e. in each $s \in S^*(X, T)$), for a given triangulation $T = (K, \theta)$ of X , by Rademacher's theorem for Lipschitz functions on Euclidean domains, applied to u expressed in Euclidean coordinates in the interior of $\theta^{-1}(s)$.

On geometrical and topological properties. Here we collect some geometrical and topological properties of Riemannian polyhedra that we need afterwards. For the Riemannian polyhedron (X, g, d_X) we have,

1. The concepts of orientation and degree of mappings make sense for circuits, also $X \setminus S$ induce the same orientation from X .
2. If X is circuit then it is non-branching.
3. A polyhedron is not necessarily an Alexandrov space (Alexandrov curvature, $\kappa > -\infty$). For example for the pinched torus we have $\kappa \geq -\infty$.
4. We call a pseudomanifold (X, g, d_X) a Lipschitz pseudomanifold, if g is Lipschitz continuous. If g is simplexwise smooth such that $(X \setminus S, g|_{X \setminus S})$ has the structure of a smooth Riemannian manifold, we call (X, g, d_X) a smooth pseudomanifold.

2.2. The Sobolev space $W^{1,2}(X)$. Let (X, g, d_X) denote an admissible Riemannian polyhedron of dimension n . We denote by $\text{Lip}^{1,2}(X)$ the linear space of all Lipschitz continuous functions $u : (X, d_X) \rightarrow \mathbb{R}$ for which the Sobolev $(1, 2)$ -norm $\|u\|$ defined by

$$\|u\|^2 = \int_X (u^2 + |\nabla u|^2) = \sum_{s \in S^{(n)}(X)} \int_s (u^2 + |\nabla u|^2)$$

is finite, $S^{(n)}(X)$ denoting the collection of all n -simplexes s of X (relative to a given triangulation), and $|\nabla u|$ the Riemannian length of the Riemannian gradient on each s . Here and elsewhere, the integration is with respect to the Riemannian volume measure on X , respectively on each s in $S^{(n)}(X)$, unless otherwise specified.

The Lebesgue space $L^2(X) = \bigoplus_{s \in S^{(n)}(X)} L^2(s)$ is likewise formed with respect to the volume measure.

The Sobolev space $W^{1,2}(X)$ is defined as the completion of $\text{Lip}^{1,2}(X)$ with respect to the above Sobolev norm $\|\cdot\|$. We use the notions $\text{Lip}_c(X)$, $W_0^{1,2}(X)$ and $W_{\text{loc}}^{1,2}(X)$, in order for the linear space of functions in $\text{Lip}(X)$ with compact support, the closure of $\text{Lip}_c(X)$ in $W^{1,2}(X)$ and all $u \in L_{\text{loc}}^2(X)$ such that $u \in W^{1,2}(U)$ for all relatively compact subdomains U in X .

Sobolev space on metric spaces. For the definitions and theorems in this part, we refer the reader to [Che99]. Let (Y, d, μ) be a metric measure space, μ Borel regular. Assume also the measure of balls of finite and non-zero radius are finite and non-zero. Fix a set $A \subset Y$. Let f be a function on A with values in the extended real numbers.

Definition 2.9. *An upper gradient, for f is an extended real valued Borel function, $g : A \rightarrow [0, \infty]$, such that for all points, $y_1, y_2 \in A$ and all continuous rectifiable curves, $c : [0, l] \rightarrow A$, parameterized by arc length s , with $c(0) = y_1$, $c(l) = y_2$, we have*

$$|f(y_2) - f(y_1)| \leq \int_0^l g(c(s)) \, ds$$

Note that in above definition the left-hand side is interpreted as ∞ , if either $f(y_1) = \pm\infty$ or $f(y_2) = \pm\infty$. If on the other hand, the right-hand side is finite then it follows that $f(c(s))$ is a continuous function of s . For a Lipschitz function f we define the lower pointwise Lipschitz constant of f at x as

$$\text{lip } f(x) = \liminf_{r \rightarrow 0} \sup_{y \in B(x, r)} \frac{|f(y) - f(x)|}{r}$$

$\text{lip } f$ is Borel, finite and bounded by the Lipschitz constant. Also $\text{lip } f$ is if upper gradient for f . Similarly for Lipschitz function f , the upper pointwise Lipschitz constant f , $\text{Lip } f$, is the Borel function

$$\text{Lip } f(x) = \limsup_{r \rightarrow 0} \sup_{y \in B(x, r)} \frac{|f(y) - f(x)|}{r}.$$

For any Lipschitz function f we have $\text{lip } f(x) \leq \text{Lip } f$. In the special case $Y = \mathbb{R}^n$, if x is a point of differentiability of f , we observe that $\text{lip } f(x) = \text{Lip } f(x) = |\nabla f(x)|$. We now define the Sobolev space $H^{1,p}$, for $1 \leq p < \infty$.

Definition 2.10. *Whenever $f \in L^p(Y)$, let*

$$\|f\|_{1,p} = \|f\|_{L^p} + \inf_{g_i} \liminf_{i \rightarrow \infty} \|g_i\|_{L^p},$$

where the infimum is taken over all sequence $\{g_i\}$, for which there exists a sequence $f_i \xrightarrow{L^p} f$, such that g_i is an upper gradient for f_i , for all i .

For $p \geq 1$, the Sobolev space, $H^{1,p}$, is the subspace of L^p consisting of functions, f , for which $\|f\|_{1,p} < \infty$, equipped with the norm $\|\cdot\|_{1,p}$. The space $H^{1,p}$ is complete. Next we define the notions of generalized upper and minimal upper gradients. This will allow us to give a nice interpretation of the $H_{1,p}$ norm of Sobolev functions.

Definition 2.11. *The function, $g \in L^p$ is a generalized upper gradient for $f \in L^p$, if there exist sequences, $f_i \xrightarrow{L^p} f$, $g_i \xrightarrow{L^p} g$, such that g_i is an upper gradient for f_i , for all i .*

Definition 2.12. *For fixed p , a minimal generalized upper gradient for f is a generalized upper gradient g_f , such that $\|f\|_{1,p} = \|f\|_{L^p} + \|g_f\|_{L^p}$.*

Theorem 2.13. *For all $1 < p < \infty$ and $f \in H^{1,p}$ there exists a minimal generalized upper gradient, g_f , which is unique up to modification on subsets of measure zero.*

We will discuss two important properties of metric spaces called the *ball doubling property* and the *Poincaré inequality* for functions on them. These are essential assumptions to get a richer theory on metric spaces.

Definition 2.14. *Let (Y, d, μ) be a metric measure space. The measure μ is said to be doubling (locally) if for all $r' > 0$ there exists $\kappa = \kappa(r')$ such that for all $y \in Y$ and $0 < r < r'$*

$$0 < \mu(B_r(y)) \leq 2^\kappa \mu(B_{r/2}(y)). \quad (2)$$

Definition 2.15. *Let $q \geq 1$. We say that Y supports a weak Poincaré inequality of type (q, p) , if for all $r' > 0$, there exist constants $1 \leq \lambda < \infty$ and $C = C(p, r') > 0$ such that for all $r \leq r'$, and all upper gradients g of f ,*

$$\left(\int_{B_r(x)} |f - f_{x,r}|^q d\mu \right)^{1/q} \leq Cr \left(\int_{\lambda B_r(x)} |g|^p d\mu \right)^{1/p}, \quad (3)$$

where $f_{x,r} := \int_{B_r(x)} f d\mu$. If $\lambda = 1$, then we say that X supports a strong (q, p) -Poincaré inequality.

Every admissible Riemannian polyhedron (X, g, μ_g) , μ_g , and hence the metric space X is doubling. Moreover X supports a weak $(2, 2)$ -Poincaré inequality and by Holder's inequality $(1, 2)$ -Poincaré inequality (see Corollary 4.1 and Theorem (5.1) in [EF01]). By abuse of notation we occasionally write " X satisfies the Poincaré inequality, to mean that X supports a Poincaré inequality of type". $(2, 2)$.

By Theorem 4.24 in [Che99], for any metric space which satisfies (2) and (3), for some $1 \leq p < \infty$ and $q = 1$, the subspace of locally Lipschitz functions is dense in $H^{1,p}$. Also by Theorem 6.1 there, on a locally complete metric space with the mentioned properties, we have for some $1 < p < \infty$ and for any $f \in H^{1,p}$, $g_f = \text{Lip } f$, μ -almost everywhere. Therefore, on a

Riemannian polyhedron (X, g, μ_g) , for any $f \in H^{1,2}$, $g_f(y) = |\nabla f(y)|$ for a.e. y and

$$H^{1,2} = W^{1,2}$$

In the following, we always consider $X = (X, g, \mu_g)$ to be an admissible Riemannian polyhedron. Some of the concepts below are defined on metric spaces in general but for simplicity we present them only on Riemannian polyhedron and for $p = 2$. For more information on metric spaces we refer the reader to [BB11].

2.3. Capacities.

Definition 2.16. [BB11] *The capacity of a set $E \subset X$ is the number*

$$C(E) = \inf \|u\|_{W^{1,2}(X)}^2,$$

where the infimum is taken over all $u \in W^{1,2}(X)$ such that $u \geq 1$ on E .

Definition 2.17. [BB11] *Assume $\Omega \subset X$ is bounded. Let $E \subset \Omega$. We define the variational capacity*

$$\text{cap}(E, \Omega) = \inf_u \int_{\Omega} |\nabla u|^2 d\mu_g, \quad (4)$$

where the infimum is taken over all $u \in W_0^{1,2}(\Omega)$ such that $u \geq 1$ on E .

In the above definitions the infimum can be taken only over $u \leq 1$ such that it is equal 1 on a neighborhood of E . Also we write $\text{cap}(E) = \text{cap}(E, X)$.

Definition 2.18. [EF01] *A set $U \subset X$ is quasi open if there are open sets ω of arbitrarily small capacity such that $U \setminus \omega$ is open relative to $X \setminus \omega$.*

Definition 2.19. [EF01] *a map $\phi : U \rightarrow Y$ from a quasiopen set U to a topological space Y with a countable base of open sets is quasicontinuous if there are open sets ω of arbitrarily small capacity such that $\phi|_{U \setminus \omega}$ is continuous*

Clearly this amounts to $\phi^{-1}(V)$ being quasiopen for every open subset V of Y .

2.4. Weakly harmonic and weakly sub/super harmonic functions.

Definition 2.20. *function $u \in W_{loc}^{1,2}(X)$ is said to be weakly harmonic if*

$$\int_X \langle \nabla u, \nabla \rho \rangle d\mu_g = 0 \quad \text{for every } \rho \in \text{Lip}_c(X).$$

Definition 2.21. *A function $u \in W_{loc}^{1,2}(X)$ is said to be weakly subharmonic, resp. weakly superharmonic, if*

$$\int_X \langle \nabla u, \nabla \rho \rangle d\mu_g \leq 0, \text{ resp. } \geq 0 \quad \text{for every } \rho \in \text{Lip}_c(X).$$

Proposition 2.22. *A function $u \in W^{1,2}(X)$ is weakly harmonic if and only if u minimizes the energy $E(v)$ among all functions $v \in W^{1,2}(X)$ such that $v - u \in W_0^{1,2}(X)$.*

Theorem 2.23. *(The variational Dirichlet problem) Suppose the following Poincaré inequality holds:*

$$\int_X |u|^2 d\mu_g \leq c \int_X |\nabla u|^2 d\mu_g \quad \text{for all } u \in W_0^{1,2}(X) \quad (5)$$

with c depending only on the admissible Riemannian polyhedron X . For any $f \in W^{1,2}(X)$ the class of competing maps

$$W_f^{1,2}(X) = \{v \in W^{1,2}(X) : v - f \in W_0^{1,2}(X)\} \quad (6)$$

contains a unique weakly harmonic function u . That function is the unique solution u of the equation $E(u) = E_0$, where

$$E_0 := \inf\{E(v) : v \in W^{1,2}(X), v - f \in \text{Lip}_c(X)\} = \min\{E(v) : v \in W_f^{1,2}(X)\}.$$

Corollary 2.24. *Assume that Ω is bounded and such that $C_p(X \setminus \Omega) > 0$. For any $f \in W^{1,2}(\Omega)$, the class of functions*

$$W_f^{1,2}(\Omega) = \{v \in W^{1,2}(\Omega) : v - f \in W_0^{1,2}(\Omega)\}$$

has a unique solution u of the equation $E(u) = E_\Omega$, where

$$E_\Omega := \inf\{E(v) : v \in W^{1,2}(\Omega), v - f \in W_0^{1,2}(\Omega)\}.$$

Proof. Since X satisfies the Poincaré inequality and using Theorem 5.54 in [BB11], Ω satisfies the inequality (5). By the theorem above, there is a unique minimizer which is weakly harmonic. \square

Note that if X is unbounded and Ω bounded then $C_p(X \setminus \Omega) > 0$ is provided. Using admissible Riemannian polyhedron satisfy the ball doubling property and the Poincaré inequality,

Theorem 2.25. *Every weakly harmonic function on X is Hölder continuous (after correction on a null set).*

A continuous weakly harmonic function on X is called harmonic.

Remark 1. *From the discussion above one can see in the definition of variational capacity that there is a harmonic function u which takes the minimum in (4). This function is not necessarily continuous on the boundary of $\Omega \setminus E$.*

2.5. Polar sets.

Definition 2.26. *A set $S \subset X$ is said to be a polar set for the capacity if for every pair of relatively compact open sets $U_1 \Subset U_2 \subset X$ such that $d(U_1, X \setminus U_2) > 0$ we have*

$$\text{cap}(E \cap \overline{U_1}, U_2) = 0.$$

According to Theorem 9.52 in [BB11] (see also Section 3 in [GT02]), S is a polar set if and only if every point of X has an open neighborhood U on which there is a superharmonic function which equals ∞ at every point of $S \cap U$. An equivalent formulation is to say that $C_2(S) = 0$

Lemma 2.27. *A closed set $S \subset X$ is a polar set if and only if for every neighborhood U of S and every $\epsilon > 0$, there exists a function $\text{Lip}(X)$ such that*

- i) the support of φ is contained in $X \setminus S$;*
- ii) $0 \leq \varphi \leq 1$;*
- iii) $\varphi \equiv 1$ on $X \setminus U$;*
- iv) $\int_X |\nabla \varphi|^p < \epsilon$.*

Proof. The proof is based on the definition of polar set and it is completely the same as the case of Riemannian manifolds. See Proposition 3.1 in [Tro99] for the proof of the equivalence on Riemannian manifolds. \square

2.6. The Dirichlet space $L_0^{1,2}(X)$. In the present setting, the Dirichlet space on the admissible Riemannian polyhedron X will be the following Hilbert space $L_0^{1,2}(X)$, a subspace of $W_{\text{loc}}^{1,2}(X)$, and with the energy norm $E(u)^{1/2}$.

Proposition 2.28. *Suppose that, for every compact set $K \subset X$,*

$$\left(\int_K |u| \, d\mu_g \right)^2 \leq c(K) E(u) \quad \text{for all } u \in \text{Lip}_c(X), \quad (7)$$

with $c(K)$ depending only on X and K . In particular, X is noncompact. The completion $L_0^{1,2}(X)$ of space $\text{Lip}_c(X)$ within $L_{\text{loc}}^1(X)$ with respect to the norm $E(u)^{1/2}$ is then a regular Dirichlet space of strongly local type. $L_0^{1,2}(X)$ is a subset of $W_{\text{loc}}^{1,2}(X)$. Also, if (X, d_X) is complete then $L_0^{1,2}(X)$ is an admissible Dirichlet space.

Note that $W_0^{1,2}(X) \subset L_0^{1,2}(X)$. According to the above proposition, $(L_0^{1,2}(X), E)$ is a strongly local regular Dirichlet form. Let

$$\Delta : L_0^{1,2}(X) \supset D(\Delta) \rightarrow L^2(X)$$

denote the generator induced from $(E, L_0^{1,2}(X))$, which is a densely defined non-positive definite self-adjoint operator satisfying $E(u, v) = (\Delta u, v)_{L^2}$. Here $D(\Delta)$ denotes the domain of operator Δ and $\text{Lip}_c(X)$ is the core of $L_0^{1,2}(X)$.

By considering Dirichlet structure $L_0^{1,2}(X)$, one can show that there is a symmetric Green function on an admissible Riemannian polyhedron and prove the following (see Proposition 7.6 in [EF01]),

Proposition 2.29. *The $(n-2)$ -skeleton $X^{(n-2)}$ of an admissible Riemannian n -polyhedron is a polar set.*

We should note that being polar is independent of the Riemannian structure on the polyhedron. Also for local questions, condition (7) is not required (it is automatically satisfied with X replaced by the open star of a point a of X relative to a sufficiently fine triangulation and in view of inequality (3)).

Remark 2. *Every closed polar subset F of X is removable for Sobolev $(1,2)$ -functions, i.e. $W^{1,2}(X \setminus F) = W^{1,2}(X)$. A larger class of removable sets in this sense is that of all (closed) sets of $(n-1)$ -dimensional Hausdorff measure zero (see Proposition 7.7 in [EF01]).*

2.7. Harmonic maps on Riemannian polyhedra. The energy of maps of a Riemannian domain into an arbitrary metric space was defined and investigated by Korevaar and Schoen [KS93]. Here, we give an introduction to the concept of energy of maps, energy minimizing maps and harmonic maps on Riemannian polyhedra. In the case that the target Y is a Riemannian C^l -manifold the energy of a map is given by the expected explicit expression (similarly when the target is a Riemannian polyhedron with continuous Riemannian metric).

Let (X, g) be an admissible n -dimensional Riemannian Polyhedron with simplexwise smooth Riemannian metric. We do not require that g is continuous across lower dimensional simplexes. Let Y be an arbitrary metric space. Denote by $L_{\text{loc}}^2(X, Y)$ the space of all μ_g -measurable maps $\varphi : X \rightarrow Y$ having separable essential range, and for which $d_Y(\varphi(\cdot), q) \in L_{\text{loc}}^2(X, \mu_g)$ for some point q (and therefore by the triangle inequality for any $q \in Y$). For $\varphi, \psi \in L_{\text{loc}}^2(X, Y)$ define their distance

$$D^2(\varphi, \psi) = \int_X d_Y^2(\varphi(x), \psi(x)) \, d\mu_g(x).$$

The approximate energy density of a map $\varphi \in L_{\text{loc}}^2(X, Y)$ is defined for $\varepsilon > 0$ by

$$e_\varepsilon(\varphi)(x) = \int_{B(x, \varepsilon)} \frac{d_Y^2(\varphi(x), \varphi(x'))}{\varepsilon^{n+2}} \, d\mu_g(x')$$

The function $e_\varepsilon(\varphi) \geq 0$ is of class $L_{\text{loc}}^1(X, \mu_g)$, [KS93].

Definition 2.30. *The energy $E(\varphi)$ of a map φ of class $L_{\text{loc}}^2(X, Y)$ is*

$$E(\varphi) = \sup_{f \in C_c(X, [0,1])} \left(\limsup_{\varepsilon \rightarrow 0} \int_X f e_\varepsilon(\varphi) \, d\mu_g \right).$$

We say that φ is locally of finite energy, and write $\varphi \in W_{\text{loc}}^{1,2}(X, Y)$, if $E(\varphi|_U) < \infty$ for every relatively compact domain $U \subset X$. For example every Lipschitz continuous map $\varphi : X \rightarrow Y$ is in $W_{\text{loc}}^{1,2}(X, Y)$. Now we give a necessary and sufficient condition for a map φ be in $\varphi \in W_{\text{loc}}^{1,2}(X, Y)$.

Lemma 2.31. *Let (X, g) be an admissible n -dimensional Riemannian polyhedron with simplexwise smooth Riemannian metric, and (Y, d_Y) a metric*

space. A map $\varphi \in L^2_{\text{loc}}(X, Y)$ is locally of finite energy if there is a function $e(\varphi) \in L^1_{\text{loc}}(X)$ such that $e_\varepsilon(\varphi) \rightarrow e(\varphi)$ as $\varepsilon \rightarrow 0$, in the sense of weak convergence of measures:

$$\lim_{\varepsilon \rightarrow 0} \int_X f e_\varepsilon(\varphi) d\mu_g = \int_X f e(\varphi) d\mu_g \quad f \in C_c(X)$$

Energy of maps into Riemannian manifolds. Let the domain be an arbitrary admissible Riemannian polyhedron (X, g) (g is only measurable with local elliptic bounds, unless otherwise specified), and the target is a Riemannian C^1 -manifold (N, h) without boundary, $\dim X = n$, $\dim Y = m$. A chart $\eta : V \rightarrow \mathbb{R}^m$ is bi-Lipschitz if the components $h_{\alpha\beta}$ of $h|_V$ have elliptic bounds:

$$\Lambda_V^{-2} \sum_{\alpha=1}^m (\eta^\alpha)^2 \leq h_{\alpha\beta} \eta^\alpha \eta^\beta \leq \Lambda_V^2 \sum_{\alpha=1}^m (\eta^\alpha)^2. \quad (8)$$

Definition 2.32. Relative to a given countable atlas on a Riemannian C^1 -manifold (N, h) , a map $\varphi : (X, g) \rightarrow (N, h)$ is of class $W^{1,2}_{\text{loc}}(X, N)$, or locally of finite energy, if

- (i) φ has a quasicontinuous version,
- (ii) its components $\varphi_1, \dots, \varphi_m$ in charts $\eta : V \rightarrow \mathbb{R}^m$ are of class $W^{1,2}(U)$ for every quasiopen set $U \subset \varphi^{-1}(V)$ of compact closure in X , and
- (iii) the energy density $e(\varphi)$ of φ , defined a.e. in each of the quasiopen sets $\varphi^{-1}(V)$ covering X by

$$e(\varphi) = (h_{\alpha\beta} \circ \varphi) \langle \nabla \varphi^\alpha, \nabla \varphi^\beta \rangle,$$

is locally integrable over (X, μ_g) .

The energy of $\varphi \in W^{1,2}_{\text{loc}}(X, N)$ is defined by $E(\varphi) = \int_X e(\varphi) d\mu_g$. There is also corresponding definition for the energy of maps into Riemannian polyhedra. There, (X, g) is admissible, $\dim X = n$, and g is measurable with elliptic bounds on each n -simplex of X . The polyhedron Y is not required to be admissible, but its Riemannian metric h is assumed to be continuous.

Energy minimizing maps. We suppose that (X, g) , n -dimensional admissible Riemannian polyhedra with g simplexwise smooth and Y any metric space.

Definition 2.33. A map $\phi \in W^{1,2}_{\text{loc}}(X, Y)$ is said to be locally E -minimizing if X can be covered by relatively compact domains $U \subset X$ for which $E(\phi|_U) \leq E(\psi|_U)$ for every map $\psi \in W^{1,2}_{\text{loc}}(X, Y)$ such that $\phi = \psi$ a.e. in $X \setminus U$.

Some of the results concerning energy minimizing maps on Riemannian manifolds, extend to the case of Riemannian polyhedra with some restrictions on the geometry of the target.

Theorem 2.34. If Y is a simply connected complete Riemannian polyhedron of non positive curvature, every locally energy minimizing map $\varphi : X \rightarrow Y$ is Holder continuous.

Theorem 2.35. *Let X and Y be compact Riemannian polyhedra. Assume that*

- (i) X is admissible, and
- (ii) Y has non positive curvature.

Then every homotopy class \mathcal{H} of continuous maps $X \rightarrow Y$ has an E -minimizer relative to \mathcal{H} , and any such is Holder continuous.

Harmonic maps. Consider an admissible Riemannian polyhedron (X, g) , $\dim X = n$, and a metric space (Y, d_Y)

Definition 2.36. *A harmonic map $\varphi : X \rightarrow Y$ is a continuous map of class $\varphi \in W_{\text{loc}}^{1,2}(X, Y)$, which is bi-locally energy minimizing in the sense that X can be covered by relatively compact subdomains U , for each of which there is an open set $V \supset \varphi(U)$ in Y such that*

$$E(\varphi|_U) \leq E(\psi|_U)$$

for every continuous map $\psi \in W_{\text{loc}}^{1,2}(X, Y)$ with $\psi(U) \subset V$ and $\varphi = \psi$ in $X \setminus U$.

Every continuous, locally energy minimizing map $\varphi : X \rightarrow Y$ is harmonic. Also if Y is a simply connected complete Riemannian polyhedron of non-positive curvature, then a harmonic map $\varphi : X \rightarrow Y$ is the same as a continuous locally energy minimizing map.

Example 2.37. *Let K be a compact group of isometries of smooth Riemannian manifolds (M, g) , (N, h) , and $\psi : M \rightarrow N$ a smooth K -equivariant map (i.e. $\psi(k \cdot x) = k \cdot \psi(x)$ for all $x \in M$, $k \in K$). Then ψ induces a continuous map $\varphi : X = M/K \rightarrow Y = N/K$. Under various circumstances, harmonicity of ψ insures that of φ (in the sense of above definition), and conversely.*

For the definition of the energy of a map, we consider the case when (X, g) is an arbitrary admissible Riemannian polyhedron, g just bounded measurable with local elliptic bounds, $\dim X = n$, and (N, h) a smooth Riemannian manifold without boundary, $\dim N = m$. We denote by $\Gamma_{\alpha\beta}^k$ the Christoffel symbols on N .

Definition 2.38. *A weakly harmonic map $\varphi : X \rightarrow N$ is a quasicontinuous map of class $W_{\text{loc}}^{1,2}(X, N)$ with the following property: for any chart $\eta : V \rightarrow \mathbb{R}^n$ on N and any quasiopen set $U \subset \varphi^{-1}(V)$ of compact closure in X , the equation*

$$\int_U \langle \nabla \lambda, \nabla \varphi^k \rangle d\mu_g = \int_U \lambda \cdot (\Gamma_{\alpha\beta}^k \circ \varphi) \langle \nabla \varphi^\alpha, \nabla \varphi^\beta \rangle d\mu_g \quad (9)$$

holds for every $k = 1, \dots, m$ and every bounded function $\lambda \in W_0^{1,2}(U)$.

A continuous map $\varphi \in W_{\text{loc}}^{1,2}(X, N)$ is harmonic (Definition 2.36) if and only if it is weakly harmonic (Definition 2.38).

3. RICCI CURVATURE ON RIEMANNIAN POLYHEDRA

In the past few years, several notions of boundedness of Ricci curvature from below on general metric spaces have appeared. Sturm [Stu06] and Lott-Villani [LV09] independently introduced the so called curvature-dimension condition on a metric measure space denoted by $CD(k, N)$. The curvature dimension condition implies the generalized Brunn-Minkowski inequality (hence the Bishop-Gromov comparison and Bonnet-Myer's theorem) and a Poincaré inequality (see [Stu06, LV07, LV09]). Meanwhile, Sturm and Ohta introduced a measure contraction property denoted by $MCP(k, N)$. The condition $MCP(k, N)$ also implies the Bishop-Gromov comparison, Bonnet-Myer's theorem and a Poincaré inequality (see [Stu06, Oht07]). Note that all of these generalized notions of Ricci curvature bounded below are equivalent to the classical one on smooth Riemannian manifolds. Here we define both conditions and show then that on a Riemannian polyhedron we can use both of them. In the following definitions, we always assume that (X, d) is a separable length space, $P(X)$ is the set of all Borel probability measures μ satisfying $\int_X d_X(x, y)^2 d\mu(y) < \infty$ for some $x \in X$. $P_2(X)$ is the set $P(X)$ equipped with the L^2 -Wasserstein distance W_2 .

Curvature Dimension Condition $CD(K, N)$. [LV09] Suppose (X, d) is a compact length space. Let $U : [0, \infty) \rightarrow \mathbb{R}$ be a continuous convex function with $U(0) = 0$. We define the non negative function

$$p(r) = rU'_+(r) - U(r)$$

with $p(0) = 0$. Given a reference probability measure $\nu \in P_2(X)$, define the function $U_\nu : P_2(X) \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$U_\nu(\mu) = \int_X U(\rho(x)) d\nu(x) + U'(\infty)\mu_s(X),$$

where

$$\mu = \rho\nu + \mu_s$$

is the Lebesgue decomposition of μ with respect to ν into an absolutely continuous part $\rho\nu$ and a singular part μ_s , and

$$U'(\infty) = \lim_{r \rightarrow \infty} \frac{U(r)}{r}.$$

If $N \in [1, \infty)$ then we define \mathcal{DC}_N to be the set of such functions U so that the function

$$\psi(\lambda) = \lambda^N U(\lambda^{-N})$$

is convex on $(0, \infty)$. We further define \mathcal{DC}_∞ to be the set of such functions U so that the function

$$\psi(\lambda) = e^\lambda U(e^{-\lambda})$$

is convex on $(-\infty, \infty)$. A relevant example of an element of \mathcal{DC}_N is given by

$$H_{N,\nu} = \begin{cases} Nr(1 - r^{-1/N}) & \text{if } 1 < N < \infty, \\ r \log r & \text{if } N = \infty. \end{cases} \quad (10)$$

Definition 3.1. *Given $N \in [1, \infty]$, we say that a compact measured length space (X, d, ν) has nonnegative N -Ricci curvature if for all $\mu_0, \mu_1 \in P_2(X)$ with $\text{supp}(\mu_0) \subset \text{supp}(\nu)$ and $\text{supp}(\mu_1) \subset \text{supp}(\nu)$, there is some Wasserstein geodesic $\{\mu_t\}_{t \in [0,1]}$ from μ_0 to μ_1 so that for all $U \in \mathcal{DC}_N$ and all $t \in [0, 1]$,*

$$U_\nu(\mu_t) \leq tU_\nu(\mu_1) + (1-t)U_\nu(\mu_0). \quad (11)$$

Given $K \in \mathbb{R}$, we say that (X, d, ν) has ∞ -Ricci curvature bounded below by K if for all $\mu_0, \mu_1 \in P_2(X)$ with $\text{supp}(\mu_0) \subset \text{supp}(\nu)$ and $\text{supp}(\mu_1) \subset \text{supp}(\nu)$, there is some Wasserstein geodesic $\{\mu_t\}_{t \in [0,1]}$ from μ_0 to μ_1 so that for all $U \in \mathcal{DC}_\infty$ and all $t \in [0, 1]$,

$$U_\nu(\mu_t) \leq tU_\nu(\mu_1) + (1-t)U_\nu(\mu_0) - \frac{1}{2}\lambda(U)t(1-t)W_2(\mu_0, \mu_1)^2, \quad (12)$$

where $\lambda : \mathcal{DC}_\infty \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined as,

$$\lambda(U) = \inf_{r>0} K \frac{p(r)}{r} = \begin{cases} K \lim_{r \rightarrow 0^+} \frac{p(r)}{r} & \text{if } K > 0, \\ 0 & \text{if } K = 0, \\ K \lim_{r \rightarrow \infty} \frac{p(r)}{r} & \text{if } K < 0. \end{cases} \quad (13)$$

Note that inequalities (11) and (12) are only assumed to hold along some Wasserstein geodesic from μ_0 to μ_1 , and not necessarily along all such geodesics. This is what is called weak displacement convexity.

Proposition 3.2. *If a compact measured length space (X, d, ν) has nonnegative N -Ricci curvature for some $N \in [1, \infty)$, then for all $x \in \text{supp}(\nu)$ and all $0 < r_1 \leq r_2$*

$$\nu(B_{r_2}(x)) \leq \left(\frac{r_2}{r_1}\right)^N \nu(B_{r_1}(x)).$$

To generalize the notion of N -Ricci curvature to the non-compact case, we always consider a complete pointed locally compact metric measure space (X, \star, ν) . Also for U_ν to be a well-defined functional on $P_2(X)$, we impose the restriction $\nu \in M_{-2(N-1)}$, where $M_{-2(N-1)}$ is the space of all non-negative Radon measures ν on X such that

$$\int_X (1 + d(\star, x)^2)^{-(N-1)} d\nu(x) < \infty.$$

We define $M_{-\infty}$, by the condition $\int_X e^{-cd(x, \star)^2} d\nu(x) < \infty$, where c is a fixed positive constant. We should mention that most of the results for compact case (for example the Bishop-Gromov comparison) are valid for the non-compact case.

Measure Contraction Property $MCP(K, N)$. [Oht07] Let (X, d_X) be a length space, and μ a Borel measure on X such that $0 < \mu(B(x, r)) < \infty$ for every $x \in X$ and $r > 0$, where $B(x, r)$ denotes the open ball with center $x \in X$ and radius $r > 0$.

Let Γ be the set of minimal geodesics, $\gamma : [0, 1] \rightarrow X$, and define the evaluation map e_t by $e_t(\gamma) := \gamma(t)$ for each $t \in [0, 1]$. We regard Γ as a subset of the set of Lipschitz maps $\text{Lip}([0, 1], X)$ with the uniform topology. A dynamical transference plan Π is a Borel probability measure on Γ , and a path $\{\mu_t\}_{t \in [0, 1]} \subset P_2(X)$ given by $\mu_t = (e_t)_* \Pi$ is called a displacement interpolation associated to Π . For $K \in \mathbb{R}$, we define the function s_K on $[0, \infty)$ (on $[0, \pi/\sqrt{K})$ if $K > 0$) by

$$s_K(t) := \begin{cases} (1/\sqrt{K}) \sin(\sqrt{K}t) & \text{if } K > 0, \\ t & \text{if } K = 0, \\ (1/\sqrt{-K}) \sinh(\sqrt{-K}t) & \text{if } K < 0. \end{cases} \quad (14)$$

Definition 3.3. For $K, N \in \mathbb{R}$ with $N > 1$, or with $K \leq 0$ and $N = 1$, a metric measure space (X, μ) is said to satisfy the (K, N) -measure contraction property (the (K, N) -MCP for short) if, for every point $x \in X$ and measurable set $A \subset X$ (provided that $A \subset B(x, \pi\sqrt{(N-1)/K})$ if $K > 0$) with $0 < \mu(A) < \infty$, there exists a displacement interpolation $\{\mu_t\}_{t \in [0, 1]} \subset P_2(X)$ associated to a dynamical transference plan $\Pi = \Pi_{x, A}$ satisfying:

- (1) We have $\mu_0 = \delta_x$ and $\mu_1 = (\mu|_A)^-$ as measures, where we denote by $(\mu|_A)^-$ the normalization of $\mu|_A$, i.e., $(\mu|_A)^- := \mu(A)^{-1} \cdot \mu|_A$;
- (2) For every $t \in [0, 1]$,

$$d\mu \geq (e_t)_* \left(t \left\{ \frac{s_K(t l(\gamma)/\sqrt{N-1})}{s_K(l(\gamma)/\sqrt{N-1})} \right\}^{N-1} \mu(A) d\Pi(\gamma) \right)$$

holds as measures on X , where we set $0/0 = 1$ and, by convention, we read

$$\left\{ \frac{s_K(t l(\gamma)/\sqrt{N-1})}{s_K(l(\gamma)/\sqrt{N-1})} \right\}^{N-1} = 1$$

if $K \leq 0$ and $N = 1$.

Here we state two results that we are going to use in the sequel.

Proposition 3.4. Let (M, g) be an n -dimensional, complete Riemannian manifold without boundary with $n \geq 2$. Then a metric measure space (M, d_g, ν_g) satisfies the (K, n) -MCP if and only if $\text{Ric}_g \geq K$ holds.

Proposition 3.5. (Bishop-Gromov volume comparison theorem) Let (X, μ) be a metric space satisfying the (K, N) -MCP. Then, for any $x \in X$, the function

$$\mu(B(x, r)) / \left\{ \int_0^r s_K \left(\frac{s}{\sqrt{N-1}} \right)^{N-1} ds \right\}$$

is monotone non-increasing in $r \in (0, \infty)$ ($r \in (0, \pi\sqrt{\frac{N-1}{K}}$) if $K > 0$).

According to Proposition 2.8, a Riemannian polyhedrapolyhedron (X, g, μ_g) with the metric $d_X = d_X^g$ is a length space. Also for any $x, y \in X$ we have

$$e(x, y) \leq d_X^e(x, y).$$

It is easy then to show that μ_g is in $M_{-2(N-1)}$ and so on a complete Riemannian polyhedrapolyhedron we can use the notion of $CD(K, N)$. Also μ_g is Borel and by Lemma 4.4 in [EF01], for any r there exist a constant $c(r)$ such that

$$c(r)^{-1} \Lambda^{-2n} r^n \leq \mu_g(B(x, r)) \leq c(r) \Lambda^{2n} r^n$$

where $x \in X$. Therefore $0 < \mu_g(B(x, r)) < \infty$ and the notion of $MCP(K, N)$ is also applicable here, for $N \geq n$. (By Theorem 2.4.3 in [AT04], we have the Hausdorff dimension is n and by Corollary 2.7 in [Oht07] N should be greater than n .)

In the rest of this paper by $\text{Ric}_{N, \mu_g} \geq K$ we mean that (X, g, μ_g) satisfies the $MCP(K, N)$. In the following Lemma we show that any complete Riemannian polyhedron with non-negative Ricci curvature has infinite volume.

Lemma 3.6. *Let (X, g, μ_g) be a complete Riemannian polyhedrapolyhedron. If $\text{Ric}_{N, \mu_g}(X) \geq 0$, for $N \geq n$, then X has infinite volume.*

Proof. By the Bishop comparison theorem, Theorem 3.5, for $x \in X$ and all $0 < r_1 \leq r_2$,

$$\mu_g(B_{r_2}(x)) \leq \left(\frac{r_2}{r_1}\right)^N \mu_g(B_{r_1}(x))$$

Consider a geodesic ray $\gamma(t)$, such that $\gamma(0) = x$. We construct the balls $B(\gamma(t), t-1)$ and $B(\gamma(t), t+1)$ centered at $\gamma(t)$ with radius $t-1$ and $t+1$. We have

$$\frac{\mu_g(B(\gamma(0), 1)) + \mu_g(B(\gamma(t), t-1))}{\mu_g(B(\gamma(t), t-1))} \leq \frac{\mu_g(B(\gamma(t), t+1))}{\mu_g(B(\gamma(t), t-1))} \leq \left(\frac{t+1}{t-1}\right)^N,$$

and so

$$1 + \frac{\mu_g(B(\gamma(0), 1))}{\mu_g(B(\gamma(t), t-1))} \leq \left(\frac{t+1}{t-1}\right)^N.$$

Letting $t \rightarrow \infty$, we get $\mu_g(B(\gamma(t), t-1)) \rightarrow \infty$ and therefore X has infinite volume. \square

By Theorem 3.2, and since X is a complete locally compact length space, the above theorem is still valid for the case when X satisfies the non-negative N -Ricci curvature condition $CD(0, N)$, for $N \in (1, \infty)$.

Remark 3. *By Remark 5.8 in [Stu06] if (X, d, μ) satisfy $MCP(k, N)$ so does any convex set $A \subset X$. When X is a pseudomanifold, for any point $x \in X \setminus S$, there exist a closed totally convex neighborhood V around x (for*

every point in a Riemannian manifold there is a geodesic ball which is totally convex). Therefore if X satisfies $\text{Ric}_{N,\mu_g} \geq K$, so does $X \setminus S$.

4. SOME FUNCTION THEORETIC PROPERTIES ON COMPLETE RIEMANNIAN POLYHEDRA

4.1. Liouville-type Theorems for Functions. The aim of this subsection is to generalize some of the results in [Yau82] in order to prove some vanishing theorems for harmonic maps on Riemannian polyhedra. In [Yau82], Yau used the Gaffney's Stokes theorem on complete Riemannian manifolds to prove that every smooth subharmonic function with bounded $\|\nabla f\|_{L_1}$ is harmonic. He uses this fact to prove there is no non-constant L^p , $p > 1$, non-negative subharmonic function on a complete manifold. We will prove this theorem on admissible polyhedra for $p = 2$. There we do not need more than $W_{\text{loc}}^{1,2}$ -regularity for the function f .

Theorem 4.1. *Suppose (X, g) is a complete, admissible Riemannian polyhedron, and $f \in W_{\text{loc}}^{1,2}(X) \cap L^2(X)$ is a non-negative, weakly subharmonic function. Then f is constant.*

Proof. Take a non-negative Lipschitz continuous function $\rho : X \rightarrow \mathbb{R}$ such that for an arbitrary point $x \in X$ and $R > 0$,

$$\begin{aligned} 0 &\leq \rho(y) \leq 1, \\ \rho(y) &= 1 \text{ for } y \in B(x, R), \\ \rho(y) &= 0 \text{ for } y \in X \setminus B(x, 2R), \\ |\nabla \rho| &< \frac{c}{R} \text{ a.e. for some constant } c > 0. \end{aligned}$$

For example we can choose ρ as follows: if $r(y) = d(y, x)$ then $\rho(y) = \eta(\frac{r(y)}{R})$ where η is a smooth function such that $0 \leq \eta(t) \leq 1$ and $\eta(t) = 1$ for $t \leq 1$ and $\eta(t) = 0$ for $t > 2$. Since f is subharmonic and non-negative, we have

$$\begin{aligned} 0 &\geq 2 \int_X \langle \nabla(\rho^2 f), \nabla f \rangle d\mu_g \\ &= 2 \int_X \langle (\nabla \rho^2) f + (\nabla f) \rho^2, \nabla f \rangle d\mu_g \\ &= \int_X \langle \nabla \rho^2, \nabla f^2 \rangle d\mu_g + 2 \int_X \rho^2 |\nabla f|^2 d\mu_g \\ &= 4 \int_X \langle \rho \nabla \rho, f \nabla f \rangle d\mu_g + 2 \int_X \rho^2 |\nabla f|^2 d\mu_g, \end{aligned}$$

and therefore

$$\begin{aligned} 0 &\leq -2 \int_X \langle \rho \nabla \rho, f \nabla f \rangle d\mu_g - \int_X \rho^2 |\nabla f|^2 d\mu_g \\ &\leq 2 \left(\int_{B_{2R} \setminus B_R} |f \nabla \rho|^2 \right)^{\frac{1}{2}} \left(\int_{B_{2R} \setminus B_R} |\rho \nabla f|^2 \right)^{\frac{1}{2}} - \int_{B_{2R} \setminus B_R} |\rho \nabla f|^2 - \int_{B_R} |\nabla f|^2. \end{aligned}$$

Since the last line is a quadratic form which is always non-negative, we have

$$\int_{B_R} |\nabla f|^2 d\mu_g \leq \int_{B_{2R} \setminus B_R} f^2 |\nabla \rho|^2 \leq \frac{c^2}{R^2} \int_{B_{2R}} f^2 d\mu_g,$$

and so

$$\int_{B_R} |\nabla f|^2 d\mu_g \leq \frac{c^2}{R^2} \int_X f^2 d\mu_g. \quad (15)$$

Sending R to infinity and using the fact that f has finite L^2 -norm, we conclude that

$$\int_X |\nabla f|^2 d\mu_g = 0.$$

Since X is admissible, f is constant on X . (First we prove that f is constant on each maximal n -simplex S and then using the $n - 1$ -chainability of X , we prove this in the star of any vertex p of X and then by connectedness on X .) \square

In the following theorem, we show that the Laplacian of every weakly subharmonic function $f \in W_{loc}^{1,2}(X)$ on a pseudomanifold in the distributional sense is a locally finite Borel measure. This gives us a verifying of Green's formula on these spaces. We then use this theorem, to prove that a continuous weakly subharmonic function with finite $\|\nabla f\|_{L^1}$ on a complete normal circuit is harmonic. Since we do not have Bochner formula on circuits, we are not able to prove the equivalent of Proposition 2 in [Yau82] in general, but only in the special case of smooth pseudomanifolds.

Theorem 4.2. *Let (X, g, μ_g) be a n -pseudomanifold. Let f be a weakly subharmonic function in $W_{loc}^{1,2}(X)$, such that $\|\nabla f\|_{L^1}$ is finite. Then there exist a unique locally finite Borel measure \mathbf{m}_f such that*

$$\int_X h \mathbf{m}_f = - \int_X \langle \nabla f, \nabla h \rangle d\mu_g \quad \text{for all } h \in \text{Lip}_c(X).$$

Proof. We consider the Lipschitz manifold $M = X \setminus S$ and the chart $\{(U_\alpha, \psi_\alpha)\}$ on M . We show that

$$\Lambda_\alpha(h) = - \int_{U_\alpha} \langle \nabla f, \nabla h \circ \psi_\alpha \rangle d\mu_g,$$

is a distribution on $D_\alpha = \text{Lip}_c(\psi_\alpha(U_\alpha))$ (with respect to the topology of uniform convergence on compact sets). The linearity is obvious. We have

$$\Lambda_\alpha(h) = - \int_{U_\alpha} \langle \nabla f, \nabla h \circ \psi_\alpha \rangle d\mu_g \leq \sup_{x \in U_\alpha} |\nabla h(x)| \cdot \|\nabla f\|_{L^1(U_\alpha)},$$

and so Λ_α is continuous. Actually one can easily show that the system of $\{\Lambda_\alpha\}$ determines a distribution on M . f is subharmonic and so Λ_α is a unique positive Radon measure (this comes from the fact that $\text{Lip}_c(X)$ is

dense in $C_c(X)$). It follows that there is a positive Radon measure m_α such that

$$\Lambda_\alpha(h) = \int_{U_\alpha} h \, dm_\alpha.$$

Now we consider the partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. We put $m = \sum_\alpha \rho_\alpha \psi_*(m_\alpha)$ and we define $\mathbf{m}_f(U) = m(U \setminus S)$ on each Borel set U . Obviously \mathbf{m}_f is positive and locally finite. The uniqueness comes from the uniqueness of m_α . \square

Gigli introduced the notion of Laplacian as a set of locally finite Borel measure (see Definition 4.4 in [Gig12]). There he proved that on infinitesimally Hilbertian spaces this set contains only one element ². Admissible Riemannian polyhedra are the examples of infinitesimally Hilbertian space.

Theorem 4.3. *Let (X, g, μ_g) be a complete non-compact n -pseudomanifold. Let f be a continuous weakly subharmonic belonging to $W_{loc}^{1,2}(X)$ such that $A_1 = \|\nabla f\|_{L^1}$ is finite. Then f is a harmonic function.*

Proof. To do so we consider a sequence of cut-off functions ρ_n for fixed $q \in X$ and R such that ρ_n is equal to 1 on $B(q, R)$ and its support is in $B(q, R+n)$. f is a subharmonic function which satisfies the condition of lemma above, so there is a unique Borel measure \mathbf{m}_f such that

$$0 \leq \int_X \rho_n \, d\mathbf{m}_f = - \int_X \langle \nabla \rho_n, \nabla f \rangle \, d\mu_g \leq \int_X |\nabla \rho_n| |\nabla f| \, d\mu_g \leq \frac{c}{n} A_1,$$

and

$$0 \leq \int_{B(q,R)} d\mathbf{m}_f \leq \int_X \rho_n \, d\mathbf{m}_f \leq \frac{c}{n} A_1.$$

Let h be any function in $\text{Lip}_c(X)$ with support in $B(q, R)$. We have

$$0 \leq \int_X h \, d\mathbf{m}_f \leq (\sup_X h) \frac{c}{n} A_1,$$

and tending n to infinity, we have

$$\int_X h \, d\mathbf{m}_f = - \int_X \langle \nabla h, \nabla f \rangle \, d\mu_g = 0,$$

and implying that f is harmonic. \square

Now we prove the equivalent of Proposition 2 in [Yau82]. We give here another proof of the theorem above for smooth pseudomanifolds under the extra assumption that f should have finite energy. Instead of Theorem 4.2, we goal Cheeger's Green formula on compact smooth pseudomanifolds in the proof.

²see Definition 4.18 in [Gig12] for the definition of infinitesimally Hilbertian

Theorem 4.4. *Let (X, g, μ_g) be a complete non-compact smooth pseudo-manifold. Suppose X has non-negative N -Ricci curvature, Ric_{N, μ_g} , for $N \geq n$. Let f be a continuous weakly subharmonic function belonging to $W_{\text{loc}}^{1,2}(X)$, such that both $A_1 = \|\nabla f\|_{L^1}$ and $A_2 = \|\nabla f\|_{L^2}$ are finite. Then f is constant.*

Proof. We present the proof in several steps.

Step 1. *We consider a sequence of cut-off functions ρ_n as above such that the support of ρ_n is in $B(q, R+n)$ for fixed $q \in X \setminus X^{n-2}$ and some R and ρ_n is equal to 1 on $B(q, R)$.*

Step 2. *The $(n-2)$ -skeleton in X , X^{n-2} , is a polar set. We consider, shrinking bounded neighborhoods U_j of X^{n-2} in $B(q, R+j)$, such that in each $B(q, R+j)$, we have*

$$U_j \supset U_{j+1} \supset \dots \supset \bigcap_{k=1}^{\infty} U_j.$$

By the definition of polar set, for every couple of open sets like U_j and U_{j-1} , we have $\text{cap}(X^{n-2} \cap U_j, U_{j-1}) = 0$. This means that for every $\epsilon_j = \frac{1}{j}$, there exists a function $\varphi_j \in \text{Lip}(X)$ such that $\varphi_j \equiv 1$ in a neighborhood of $X^{n-2} \cap U_j$ and ρ_j is zero outside U_{j-1} and $\int_X |\nabla \varphi_j|^2 < \frac{1}{j}$. Moreover we have $0 \leq \varphi_j \leq 1$. The function $\eta_j = 1 - \varphi_j$ has this property that the closure of its support, $\overline{\text{supp} \eta_j}$, is contained in $X \setminus X^{n-2}$ and the set $K_j = \overline{\text{supp} \eta_j} \cap \overline{B(q, R+j)}$, is a compact set. Furthermore K_j s make an exhaustion of $M = X \setminus X^{n-2}$.

Step 3. *According to Theorem 2 in [GW79], on each K_j , there exist a smooth subharmonic function f_j on M such that $\sup_{x \in K_j} |f_j(x) - f(x)| < \frac{1}{j}$ and $|\nabla f_j(x)| \leq |\nabla f(x)|$ on K_j .*

Step 4. *Let (Y, h) be a closed n -dimensional admissible Riemannian polyhedron, then for $\zeta, \psi \in \text{Dom}(\Delta)$ we have the following Stokes theorem on $Y \setminus Y^{n-2}$, (see Theorem 5.1 in [Che80])*

$$\int_{Y \setminus Y^{n-2}} \Delta \zeta \cdot \psi \, d\mu_h = - \int_{Y \setminus Y^{n-2}} \langle \nabla \zeta, \nabla \psi \rangle \, d\mu_h. \quad (16)$$

As in [Che80], every closed smooth pseudomanifold (Y, h) such that h is equivalent to some piecewise flat metric is admissible (in the sense of Cheeger).

Now suppose Y_j is an arbitrary Riemannian polyhedron containing $B(q, R+j)$. We consider its double \overline{Y}_j and equip it with a Riemannian metric \overline{g}_j , coming from the induced metric on Y_j on both copies. With this metric, \overline{Y}_j is an admissible closed Riemannian pseudomanifold. (because the metric g_j on Y_j is equivalent to piecewise flat metric g^e as it is constructed in section 2.1).

We extend ρ_j to \overline{Y}_j such that it is zero on the copy of Y_j and f_j and η_j such that they are the same functions on the copy of Y_j . The function f_j is in $W_{\text{loc}}^{1,2}(\overline{Y}_j)$. (see Theorem 1.12.3. in [KS93]).

By applying formula (16) on \overline{Y}_j , for the functions f_j and $\xi_j = \rho_j \cdot \eta_j$, we obtain

$$\int_{M_j} \Delta f_j \cdot \xi_j \, d\mu_{g_j} = - \int_{M_j} \langle \nabla f_j, \nabla \xi_j \rangle \, d\mu_{g_j},$$

where $M_j = \overline{Y}_j \setminus \overline{Y}_j^{n-2}$, since $\xi_j \in \text{Lip}_c(M) \cap Y_j$ we can write the Stokes formula above as follows

$$\int_M \Delta f_j \cdot \xi_j \, d\mu_g = - \int_M \langle \nabla f_j, \nabla \xi_j \rangle \, d\mu_g.$$

Step 5. In this step, we prove that f is harmonic on M . By use of this fact that $\text{supp}(\xi_j) \subset K_j$ we have

$$\begin{aligned} & - \int_M \langle \nabla f_j, \nabla(\rho_j \cdot \eta_j) \rangle \, d\mu_g \\ &= - \int_M |\langle \nabla f_j, \eta_j \cdot (\nabla \rho_j) \rangle| \, d\mu_g - \int_M \langle \nabla f_j, \rho_j \cdot (\nabla \eta_j) \rangle \, d\mu_g \\ &\leq \int_{K_j} |\nabla f_j| |\nabla \rho_j| \, d\mu_g + \int_{K_j} |\nabla f_j|^2 \, d\mu_g \cdot \int_{K_j} |\nabla \eta_j|^2 \, d\mu_g \\ &\leq \frac{1}{j} \int_M |\nabla f|^2 \, d\mu_g + \frac{c}{j} \int_M |\nabla f| \, d\mu_g, \end{aligned}$$

so

$$0 \leq \int_M \Delta f_j \cdot \xi_j \, d\mu_g \leq \frac{1}{j} (A_2 + cA_1). \quad (17)$$

Let h be any smooth function with compact support in $M \cap B(q, R)$. Then there is a K_m such that the support of h is in $B(q, R) \cap K_m$. For j large enough we will have $\xi_j \equiv 1$ on K_m and so we have

$$0 \leq \int_{B(q, R) \cap K_m} \Delta f_j \, d\mu_g \leq \frac{1}{j} (A_2 + cA_1).$$

considering the formula (16) as above, for j large enough we have

$$\begin{aligned} 0 \leq \int_M \Delta h \cdot f_j \, d\mu_g &= \int_M h \cdot \Delta f_j \, d\mu_g \\ &\leq (\sup h) \cdot \frac{1}{j} (A_2 + cA_1). \end{aligned}$$

Letting j go to infinity, we got $\int_M \Delta h \cdot f \, d\mu_g = 0$. By use of Weyl's lemma f is a smooth harmonic function.

Step 6. Now we show f is constant. Since M has non-negative Ricci curvature, by the Bochner formula $|\nabla f|$ is subharmonic on M and so on X . By Lemma 1.1, $|\nabla f|$ is constant. Since the L^2 -norm of $|\nabla f|$ is finite we have $|\nabla f| \equiv 0$. By Lemma 3.6, f should be constant.

□

4.2. Vanishing Results for Harmonic Maps on Complete Smooth PseudoManifolds.

Proposition 4.5. *Let (X, g, μ_g) be a complete n -dimensional Riemannian pseudomanifold. Suppose X has non-negative N -Ricci curvature, Ric_{N, μ_g} , for N finite. Suppose Y is a Riemannian manifold of non-positive curvature and $u : (X, g) \rightarrow (Y, h)$ a continuous harmonic map belonging to $W_{\text{loc}}^{1,2}(X, Y)$. If u has finite energy, then it is a constant map.*

Proof. Let $u : (X, g) \rightarrow (Y, h)$ be a continuous harmonic map in $W^{1,2}(X, Y)$. By Remark 3, we know that on the Riemannian manifold $M = X \setminus S$ we have non-negative Ricci curvature.

We show that for $\epsilon > 0$, $\sqrt{e(u) + \epsilon}$ is weakly subharmonic on X . As the restriction maps $u = u|_M : (M, g) \rightarrow Y$ is harmonic, we have a Bochner type formula for harmonic map on M and

$$\Delta e(u) > |B(u)|^2,$$

also

$$|\nabla e(u)|^2 \leq 2e(u)|B(u)|^2,$$

and so for $\epsilon > 0$, on $X \setminus S$

$$\Delta \sqrt{e(u) + \epsilon} \geq 0$$

We have $\sqrt{e(u) + \epsilon}$ in $W_{\text{loc}}^{1,2}(X, Y)$ is weakly subharmonic on X :

$$\int_X \langle \nabla \sqrt{e(u) + \epsilon}, \nabla \rho \rangle d\mu_g \leq 0 \quad \rho \in \text{Lip}_c(X).$$

As in the proof of Theorem 1.1,

$$\int_{B_R} |\nabla \sqrt{e(u) + \epsilon}|^2 d\mu_g \leq \frac{c^2}{R^2} \int_{B_{2R}} e(u) + \epsilon d\mu_g \quad (18)$$

(note that $\sqrt{e(u) + \epsilon}$ satisfies all the assumptions of the Theorem 1.1, except the finiteness of L^2 -norm which we do not need in this step).

Set $B'_R = B_R \setminus \{x \in B_R, e(u)(x) = 0\}$. Then

$$\int_{B'_R} \frac{|\nabla e(u) + \epsilon|^2}{4(e(u) + \epsilon)} d\mu_g \leq \frac{c^2}{R^2} \int_{B_{2R}} e(u) + \epsilon d\mu_g. \quad (19)$$

Letting $\epsilon \rightarrow 0$ gives

$$\int_{B'_R} \frac{|\nabla e(u)|^2}{4e(u)} d\mu_g \leq \frac{c^2}{R^2} \int_{B_{2R}} e(u) d\mu_g, \quad (20)$$

and letting $R \rightarrow \infty$ and by finiteness of the energy we have

$$\int_{B'_R} \frac{|\nabla e(u)|^2}{4e(u)} d\mu_g \leq 0, \quad (21)$$

which implies that $e(u)$ is constant. If $e(u)$ is not zero everywhere this means that the volume of X is finite. By Lemma 3.6, this is impossible and so u is constant. \square

Now we extend this theorem to the case when the target Y is a metric space. By the following lemma we have the function $d(u(\cdot), q)$, where q is an arbitrary point in Y , is subharmonic under suitable assumption on the curvature of Y . We refer the reader to [EF01] Lemma 10.2, for the proof.

Lemma 4.6. *Let (X, g) be an admissible Riemannian polyhedron, g simplexwise smooth. Let (Y, d_Y) be a simply connected complete geodesic space of non-positive curvature, and let $u \in W_{\text{loc}}^{1,2}(X, Y)$ be a locally energy minimizing map. Then u is a locally essentially bounded map and for any $q \in Y$, the function $\sqrt{u_q} = d(u(\cdot), q)$ of class $W_{\text{loc}}^{1,2}(X, Y)$ is weakly subharmonic, and in particular essentially locally bounded. Hence $u_q \in W_{\text{loc}}^{1,2}(X, Y)$ and u_q is itself weakly subharmonic.*

Now we have

Proposition 4.7. *Let (X, g, μ_g) be a complete non-compact smooth pseudo-manifold. Suppose X has non-negative N -Ricci curvature, Ric_{N, μ_g} , for N finite. Let Y be a simply connected complete geodesic space of non-positive curvature and $u : (X, g) \rightarrow Y$ a continuous harmonic map with finite energy belonging to $W_{\text{loc}}^{1,2}(X, Y)$. If $\int_M \sqrt{e(u)} d\mu_g < \infty$, then u is a constant map.*

Proof. According the lemma above the function $v(x) = d(u(x), u(x_0))$ for some $x_0 \in X$, is weakly subharmonic. We know that $|\nabla v|^2 \leq ce(u)$, where c is a constant. v is a continuous subharmonic function whose gradient is bounded by an L^1 and L^2 integrable function. According to Lemma 1.1, v is a constant function and so u is a constant map. \square

5. 2-PARABOLIC RIEMANNIAN POLYHEDRA

5.1. Definition and Properties. In this subsection we define the notion of 2-parabolicity for admissible Riemannian polyhedra. This is just the generalization of the concept of 2-parabolicity on Riemannian manifold and it coincides with the one in [GT02] (Definition 2.11) for metric measure space in general. We then prove some classical results on these spaces. We should mention that all the results which will appear in this subsection and the next subsection, except Proposition 5.5 and its corollaries are the generalizations of equivalent ones for Riemannian manifolds and the proofs are exactly the same.

Definition 5.1. *Let (X, g) be an admissible Riemannian polyhedron. Let Ω be a connected domain in X . We say Ω is 2-parabolic if for every compact set $D \subset X$ with non-empty interior, $\text{cap}(D, \Omega) = 0$. Otherwise we call (X, g) 2-hyperbolic.*

Lemma 5.2. *The domain Ω is 2-parabolic if and only if there exists a sequence of functions $\rho_j \in \text{Lip}_c(\Omega)$ such that $0 \leq \rho_j \leq 1$, ρ_j converges to 1 uniformly on every compact subset of Ω and*

$$\int_{\Omega} |\nabla \rho_j|^2 d\mu_g \rightarrow 0.$$

Proof. First suppose Ω is 2-parabolic. Then every compact set $D \subset \Omega$, with nonempty interior satisfies $\text{cap}(D, \Omega) = 0$. We choose an exhaustion $D \subset D_1 \subset D_2 \subset \dots \subset \Omega$ of Ω by compact subsets such that $\text{cap}(D_j, \Omega) = 0$ for all j . Hence we can find the function $\rho_j \in \text{Lip}_c(\Omega)$ (using the fact that $\text{Lip}_c(\Omega)$ is dense in $W_0^{1,2}(\Omega)$) such that $\rho_j \equiv 1$ on D_j and $\int_{\Omega} |\nabla \rho_j|^2 d\mu_g \leq 1/j^2$. We have constructed the desired sequence ρ_j .

Conversely, suppose there exist, a sequence $\rho_j \in \text{Lip}_c(\Omega)$ with the stated properties. Then we can find a compact subset $B \subset \Omega$ and j_0 such that $\rho_j \geq 1/2$ for every $j \geq j_0$. It follows that $\text{cap}(B, \Omega) = 0$ \square

Lemma 5.3. *If X is a 2-parabolic admissible Riemannian polyhedron and $S \subset X$ is a polar set, then $\Omega := X \setminus S$ is 2-parabolic.*

Proof. X is 2-parabolic, so by Lemma 5.2, there are an exhaustion of X and a sequence of function $\rho_j \in \text{Lip}_c(X)$ such that $0 \leq \rho_j \leq 1$ and $\rho_j \rightarrow 1$ uniformly on each compact set, and $\int_X |\nabla \rho_j|^2 d\mu_g \rightarrow 0$. Also by Lemma 2.27 there exist another sequence of functions φ_j with support in $X \setminus S$ such that $\varphi_j \rightarrow 1$ on each compact set of $X \setminus S$ and $\int_X |\nabla \varphi_j|^2 d\mu_g \rightarrow 0$. The functions $\rho_j \varphi_j$ on Ω provide the condition for 2-parabolicity in Lemma 5.2. \square

5.2. Some Function Theoretic Properties and Application to Harmonic Maps. In this subsection we prove the analogue of Theorem 1.3, for 2-parabolic admissible Riemannian polyhedra.

Proposition 5.4. *Let (X, g) be 2-parabolic admissible Riemannian polyhedron. Suppose f in $W_{\text{loc}}^{1,2}(X)$ is a positive, continuous superharmonic function on X . Then f is constant.*

Proof. Since f is continuous, for any ϵ and at any point x_0 in X there exist a relatively compact neighborhood B_0 of x_0 such that $f(x) > f(x_0) - \epsilon$ on $\overline{B_0}$ (we choose the triangulation such that x_0 is a vertex and the star of x_0 is equal to B_0). X is 2-parabolic, so $\text{cap}(B_0, X) = 0$. Consider an exhaustion of X by regular domains U_i such that $B_0 \Subset U_1 \Subset U_2 \Subset \dots \Subset X$. By Corollary 11.25 in [BB11], such exhaustion exists. There exist functions u_i which are harmonic on $U_i \setminus \overline{B_0}$, $u_i \equiv 1$ on B_0 and $u_i \equiv 0$ on $X \setminus U_i$ (see Lemma 11.17 and 11.19 in [BB11].) Also maximum principle (see Theorem 5.3 [EF01] or Lemma 10.2 [BB11] for the comparison principle). implies

$$\begin{cases} 0 \leq u_i \leq 1 \\ u_{i+1} \geq u_i \text{ on } U_i. \end{cases}$$

Define the function $h_i = (f(x_0) - \epsilon)u_i$. X is 2-parabolic, and so $\lim_{i \rightarrow \infty} h_i = f(x_0) - \epsilon$. On the other hand $f \geq h_i$ on the boundary of $U_i \setminus \overline{B_0}$. By the

comparison principle $f \geq h_i$ in $U_i \setminus \overline{B_0}$, so $f \geq f(x_0) - \epsilon$ on X . Letting $\epsilon \rightarrow 0$, we obtain $f \geq f(x_0)$. If f is non constant, there exist $x_1 \in X$ with $f(x_1) > f(x_0)$. By the same argument we obtain $f > f(x_1)$. This is a contradiction and thus f is constant. \square

Proposition 5.5. *Let X be 2-parabolic pseudomanifold. Let f in $W_{\text{loc}}^{1,2}(X)$ be a continuous weakly subharmonic function such that $\|\nabla f\|_{L^1}$ and $\|\nabla f\|_{L^2}$ are finite. Then f is harmonic.*

Proof. X is 2-parabolic so by Lemma 5.2, there is a sequence ρ_j which converges to 1 uniformly on compact sets. Suppose we construct D , D_j and ρ_j as in the proof of lemma 5.2.

$$\begin{aligned} 0 \leq - \int_X \langle \nabla \rho_j, \nabla f \rangle d\mu_g &\leq \left(\int_X |\nabla \rho_j|^2 d\mu_g \right)^{\frac{1}{2}} \left(\int_X |\nabla f|^2 d\mu_g \right)^{\frac{1}{2}} \\ &\leq \frac{1}{j} \|\nabla f\|_{L^2}^2. \end{aligned}$$

By Lemma 4.2, there is a locally finite Borel measure \mathbf{m}_f such that

$$0 < \int_D \mathbf{m}_f \leq \left| \int_X \langle \nabla \rho_j, \nabla f \rangle d\mu_g \right| \leq \frac{1}{j} \|\nabla f\|_{L^2}^2.$$

Now let h be an arbitrary test function in $\text{Lip}_c(X)$ where its support is in D . We have

$$0 \leq \int_D h \mathbf{m}_f \leq (\sup_X h) \frac{\|\nabla f\|_{L^2}^2}{j}.$$

and so f is harmonic on X . \square

Proposition 5.6. *Let X be 2-parabolic admissible Riemannian polyhedron. Let f in $W_{\text{loc}}^{1,2}(X)$ be a harmonic function such that $\|\nabla f\|_{L^2}$ is finite. Then f is constant function.*

Proof. Set

$$f_i = \max(-i, \min(i, f)).$$

Let U_i be an exhaustion of X by regular domains $U_i \subset U_{i+1} \Subset X$. There is function $u_{i,j}$ such that $u_{i,j}$ are harmonic on U_j and $u_{i,j} = f_i$ in $X \setminus U_i$. Also $u_{i,j}$ is continuous on X and $\|\nabla u_{i,j}\|_{L^2}$ is finite. We have $-i \leq u_{i,j} \leq i$. According to Theorem 6.3 in [EF01], $u_{i,j}$ are locally uniformly holder continuous, and so there is a subsequence which converges locally uniformly to some u_i . The function u_i is bounded and harmonic and hence is constant. Moreover $u_{i,j} - f_i \in L_0^{1,2}$ and so $f_i \in L_0^{1,2}$. Therefore

$$\int_X |\nabla f|^2 d\mu_g = \lim_{i \rightarrow \infty} \int_X \langle \nabla f, \nabla f_i \rangle d\mu_g = 0,$$

and f is constant. \square

Corollary 5.7. *Let (X, g) be a 2-parabolic pseudomanifold with g simplexwise smooth. Let Y be a simply connected, complete, geodesic space of non-positive curvature and $u : (X, g) \rightarrow Y$ a continuous, harmonic map with finite energy, belonging to $W_{\text{loc}}^{1,2}(X, Y)$. If we have $\int_X \sqrt{e(u)} d\mu_g < \infty$ then u is a constant map.*

Corollary 5.8. *Let (X, g) be a 2-parabolic admissible Riemannian polyhedron with simplexwise smooth metric g . Let Y be a complete, geodesic space of non-positive curvature and $u : (X, g) \rightarrow Y$ a continuous, harmonic map belonging to $W_{\text{loc}}^{1,2}(X, Y)$ with bounded image. Then u is a constant map.*

REFERENCES

- [AT04] Luigi Ambrosio and Paolo Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.
- [BB11] Anders Björn and Jana Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011.
- [Che80] Jeff Cheeger. On the Hodge theory of Riemannian pseudomanifolds. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 91–146. Amer. Math. Soc., Providence, R.I., 1980.
- [Che95] Jingyi Chen. On energy minimizing mappings between and into singular spaces. *Duke Math. J.*, 79(1):77–99, 1995.
- [Che99] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9(3):428–517, 1999.
- [DM08] Georgios Daskalopoulos and Chikako Mese. Harmonic maps from a simplicial complex and geometric rigidity. *J. Differential Geom.*, 78(2):269–293, 2008.
- [DM10] Georgios Daskalopoulos and Chikako Mese. Harmonic maps between singular spaces I. *Comm. Anal. Geom.*, 18(2):257–337, 2010.
- [EF01] J. Eells and B. Fuglede. *Harmonic maps between Riemannian polyhedra*, volume 142 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001. With a preface by M. Gromov.
- [Gig12] N. Gigli. On the differential structure of metric measure spaces and applications. *ArXiv e-prints*, May 2012.
- [GS92] Mikhail Gromov and Richard Schoen. Harmonic maps into singular spaces and p -adic superrigidity for lattices in groups of rank one. *Inst. Hautes Études Sci. Publ. Math.*, (76):165–246, 1992.
- [GT02] Vladimir Gol’dshstein and Marc Troyanov. Capacities in metric spaces. *Integral Equations Operator Theory*, 44(2):212–242, 2002.
- [GW79] R. E. Greene and H. Wu. Smooth approximations of convex, subharmonic, and plurisubharmonic functions. *Ann. Sci. École Norm. Sup. (4)*, 12(1):47–84, 1979.
- [Hil82] Stefan Hildebrandt. Liouville theorems for harmonic mappings, and an approach to Bernstein theorems. In *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 107–131. Princeton Univ. Press, Princeton, N.J., 1982.
- [Hil85] Stefan Hildebrandt. Harmonic mappings of Riemannian manifolds. In *Harmonic mappings and minimal immersions (Montecatini, 1984)*, volume 1161 of *Lecture Notes in Math.*, pages 1–117. Springer, Berlin, 1985.
- [HJW81] S. Hildebrandt, J. Jost, and K.-O. Widman. Harmonic mappings and minimal submanifolds. *Invent. Math.*, 62(2):269–298, 1980/81.

- [Hol90] Ilkka Holopainen. Nonlinear potential theory and quasiregular mappings on Riemannian manifolds. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (74):45, 1990.
- [KS93] Nicholas J. Korevaar and Richard M. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1(3-4):561–659, 1993.
- [KS01] Kazuhiro Kuwae and Takashi Shioya. On generalized measure contraction property and energy functionals over Lipschitz maps. *Potential Anal.*, 15(1-2):105–121, 2001. ICPA98 (Hammamet).
- [KS03] Kazuhiro Kuwae and Takashi Shioya. Sobolev and Dirichlet spaces over maps between metric spaces. *J. Reine Angew. Math.*, 555:39–75, 2003.
- [KS08] Kazuhiro Kuwae and Karl-Theodor Sturm. On a Liouville type theorem for harmonic maps to convex spaces via Markov chains. In *Proceedings of RIMS Workshop on Stochastic Analysis and Applications*, RIMS Kôkyûroku Bessatsu, B6, pages 177–191. Res. Inst. Math. Sci. (RIMS), Kyoto, 2008.
- [LV07] John Lott and Cédric Villani. Weak curvature conditions and functional inequalities. *J. Funct. Anal.*, 245(1):311–333, 2007.
- [LV09] John Lott and Cédric Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.
- [Oht07] Shin-ichi Ohta. On the measure contraction property of metric measure spaces. *Comment. Math. Helv.*, 82(4):805–828, 2007.
- [Stu06] Karl-Theodor Sturm. On the geometry of metric measure spaces. II. *Acta Math.*, 196(1):133–177, 2006.
- [SY76] Richard Schoen and Shing Tung Yau. Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature. *Comment. Math. Helv.*, 51(3):333–341, 1976.
- [Tro99] M. Troyanov. Parabolicity of manifolds. *Siberian Adv. Math.*, 9(4):125–150, 1999.
- [Yau82] Shing Tung Yau. Erratum: “Some function-theoretic properties of complete Riemannian manifold and their applications to geometry” [Indiana Univ. Math. J. **25** (1976), no. 7, 659–670; MR **54** #5502]. *Indiana Univ. Math. J.*, 31(4):607, 1982.

E-mail address: zahra.sinaei@epfl.ch